Closed *EP* and Hypo-*EP* Operators on Hilbert Spaces

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Notations

operators	linear operators
spaces	Hilbert spaces
A^*	adjoint of A
$A^\dagger$	Moore-Penrose inverse of A
$\mathcal{R}(A)$	range of A
$\mathcal{N}(A)$	nullspace of A
$\mathcal{B}(\mathcal{H})$	all bounded operators from $\mathcal{H}$ to itself
$\mathcal{C}(\mathcal{H})$	all closed densely defined operators from $\mathcal{H}$ to itself

Introduction

In 1950, Hans Schwerdtfeger[1] defined a new class of matrices called *EP* matrices.

Definition 1.

A square matrix A of order n with elements from the complex field $\mathbb{C}$ is called an EP matrix *if*

$$\sum_{i=1}^{n} \alpha_i A_{(i)} = 0 \text{ if and only if } \sum_{i=1}^{n} \overline{\alpha_i} A^{(i)} = 0$$

where $A_{(i)}$ is the ith row of A and $A^{(i)}$ is the ith column of A.

[1] Hans Schwerdtfeger, *Introduction to Linear Algebra and the Theory of Matrices*, P. Noordhoff, Groningen, 1950.

Literature

Pearl (1966) : A is EP iff $AA^\dagger = A^\dagger A$.

Campbell and Meyer (1975) : Let $A \in \mathcal{B}(\mathcal{H})$ have a closed range. Then A is EP iff $AA^\dagger = A^\dagger A$.

Itoh (2005) : A is hypo-EP if $A^\dagger A - AA^\dagger \geq 0$.

Meenakshi, Baksalary (EP matrices) ;

Djordjevic (EP operators) ;

Patel and Shekhawat ; Johnson and Vinoth (hypo-EP operators).

Bounded Vs Unbounded

An operator on a Hilbert space $\mathcal{H}$ is a bounded operator if and only if it is continuous. It follows that unbounded operators are discontinuous (everywhere).

Definition 2.

Let A be an operator from a Hilbert space $\mathcal{H}$ with domain $\mathcal{D}(A)$ to a Hilbert space $\mathcal{K}$. If the graph of A defined by

$$\mathcal{G}(A) = \{(x, Ax) : x \in \mathcal{D}(A)\}$$

is closed in $\mathcal{H} \times \mathcal{K}$, then A is called a closed operator.

We consider densely defined closed operators from $\mathcal{H}$ to itself.

$\mathcal{D}(A) \cap \mathcal{N}(A)^{\perp}$, the carrier of A and it is denoted by $C(A)$. We note that, for any $A \in \mathcal{C}(\mathcal{H})$, the closure of $C(A)$ is $\mathcal{N}(A)^{\perp}$.

Theorem 3 (Ben-Israel, 2003).

Let $A \in \mathcal{C}(\mathcal{H})$. Then the following are true.

1. $\mathcal{N}(A) = \mathcal{R}(A^*)^\perp, \quad \mathcal{N}(A^*A) = \mathcal{N}(A).$
2. $\mathcal{N}(A^*) = \mathcal{R}(A)^\perp, \quad \mathcal{N}(AA^*) = \mathcal{N}(A^*).$
3. $\overline{\mathcal{R}(A)} = \mathcal{N}(A^*)^\perp, \quad \overline{\mathcal{R}(A)} = \overline{\mathcal{R}(AA^*)}.$
4. $\overline{\mathcal{R}(A^*)} = \mathcal{N}(A)^\perp, \quad \overline{\mathcal{R}(A^*)} = \overline{\mathcal{R}(A^*A)}.$

Moore-Penrose Inverse

The Moore-Penrose inverse $A^\dagger$ for a closed densely defined operator A can be defined with $\mathcal{D}(A^\dagger) := \mathcal{R}(A) \oplus \mathcal{R}(A)^\perp$ and taking values in $C(A)$ by associating each $y \in \mathcal{D}(A^\dagger)$ to the unique $A^\dagger y$ such that $AA^\dagger y = Qy$, where Q is the orthogonal projection of $\mathcal{H}$ onto $\overline{\mathcal{R}(A)}$. It can be seen that $\mathcal{N}(A^\dagger) = \mathcal{R}(A)^\perp$ and

$$A^\dagger Ax = Px \text{ for } x \in \mathcal{D}(A),$$

where P is the orthogonal projection of $\mathcal{H}$ onto $\overline{C(A)}$. Again, $A^\dagger$ is closed and densely defined operator[2].

[2] M. Thamban Nair, *Linear Operator Equations: Approximations and Regularization*, World Scientific, First Edition, 2009.

Closed *EP* Operators

Definition 4.

Let A be a densely defined closed operator on a Hilbert space $\mathcal{H}$. The operator A is said to be an EP operator *if A has a closed range and $\mathcal{R}(A) = \mathcal{R}(A^*)$.*

Example 5.

Define A on ℓ_2 by $A(x_1, x_2, x_3, \ldots) = (x_1, 2x_2, 3x_3, \ldots)$ with domain $\mathcal{D}(A) = \{(x_1, x_2, x_3, \ldots) \in \ell_2 : \sum_{n=1}^{\infty} |nx_n|^2 < \infty\}$. Then $A \in \mathcal{C}(\mathcal{H})$ and it is an EP operator.

Example 6.

Define A on ℓ_2 by $A(x_1, x_2, x_3, \ldots) = \left(x_1, 2x_2, \frac{x_3}{3}, 4x_4, \frac{x_5}{5}, \ldots\right)$ with domain

$\mathcal{D}(A) = \{(x_1, x_2, x_3, \ldots) \in \ell_2 : (x_1, 2x_2, \frac{x_3}{3}, 4x_4, \frac{x_5}{5}, \ldots) \in \ell_2\}$.

Then A is not an EP operator.

Closed *EP* Operators

Theorem 7.

Let $A \in \mathcal{C}(\mathcal{H})$ with a closed range. Then the following are equivalent:

1. *A is EP ;*
2. $AA^{\dagger} = A^{\dagger}A$ *on* $\mathcal{D}(A)$;
3. $\mathcal{N}(A) = \mathcal{N}(A^{\dagger})$;
4. $\mathcal{N}(A) = \mathcal{N}(A^{*})$;
5. $\mathcal{N}(A)^{\perp} = \mathcal{R}(A)$;
6. $\overline{C(A)} = \mathcal{R}(A)$;
7. $\mathcal{H} = \mathcal{R}(A) \oplus \mathcal{N}(A)$.

Remark 8.

It is proved that that $AA^\dagger = A^\dagger A$ *on* $\mathcal{D}(A)$ *if and only if* $\mathcal{N}(A) = \mathcal{N}(A^*)$. *If we drop the assumption that* $\mathcal{R}(A)$ *is closed, we get that* $AA^\dagger \subseteq A^\dagger A$ *if and only if* $\mathcal{N}(A) = \mathcal{N}(A^*)$ *and* $\mathcal{D}(A^\dagger) \subseteq \mathcal{D}(A)$.

Similarly, we can prove that $A^\dagger A \subseteq AA^\dagger$ *if and only if* $\mathcal{N}(A) = \mathcal{N}(A^*)$ *and* $\mathcal{D}(A) \subseteq \mathcal{D}(A^\dagger)$.

Closed Hypo-*EP* Operators

Definition 9.

Let A be a densely defined closed operator on a Hilbert space $\mathcal{H}$. The operator A is said to be a hypo-*EP* operator *if A has a closed range and $\mathcal{R}(A) \subseteq \mathcal{R}(A^*)$.*

Example 10.

Define A on ℓ_2 by

$$A(x_1, x_2, x_3, \ldots) = (0, x_1, 2x_2, 3x_3, \ldots)$$

with $\mathcal{D}(A) = \left\{(x_1, x_2, \ldots) \in \mathcal{H} : \sum_{n=1}^{\infty} |nx_n|^2 < \infty\right\}$. Then A is hypo-EP but not EP.

Closed Hypo-*EP* Operators

Theorem 11.

Let $A \in \mathcal{C}(\mathcal{H})$. Then each of the following statements implies the next statement:

1. *A is hypo-EP ;*
2. $A(A^\dagger)^2A = AA^\dagger$ *on* $\mathcal{D}(A)$ *;*
3. $AA^\dagger \leq A^\dagger A$ *on* $\mathcal{D}(A)$ *;*
4. $\|AA^\dagger x\| \leq \|A^\dagger Ax\|$ *for all* $x \in \mathcal{D}(A)$.

Remark 12.

All are equivalent if $\mathcal{R}(A) \subseteq \mathcal{D}(A)$.

A perturbation result

Theorem 13.

Let $A \in \mathcal{C}(\mathcal{H})$ be an EP operator. Let $B \in \mathcal{B}(\mathcal{H})$ be such that $\|B\|\|A^\dagger\| < 1, BA^\dagger A = B|_{\mathcal{D}(A)}$ and $AA^\dagger B = B$. Then $A + B$ is EP.

Present Work :

Restriction ; Sum ; Product ; Limit ; Perturbation ;
Fuglede-Putnam Type Theorems and so on.

Main References

1. M. Thamban Nair, *Functional Analysis: A First Course*, Prentice-Hall of India, second edition, 2021.
2. Hans Schwerdtfeger, *Introduction to Linear Algebra and the Theory of Matrices*, P. Noordhoff, Groningen, 1950.
3. M. H. Pearl, *On generalized inverses of matrices*, Proc. Cambridge Philos. Soc., 62:673–677, 1966.
4. Stephen L. Campbell and Carl D. Meyer, *EP operators and generalized inverses*, Canad. Math. Bull., 18(3):327–333, 1975.
5. Masuo Itoh, *On some EP operators*, Nihonkai Math. J., 16(1):49–56, 2005.